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SOME ISOMORPHICALLY POLYHEDRAL ORLICZ SEQUENCE SPACES

BY

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ABSTRACT

A Banach space is polyhedral if the unit ball of each of its finite dimensional subspaces is a polyhedron. It is known that a polyhedral Banach space has a separable dual and is c_0-saturated, i.e., each closed infinite dimensional subspace contains an isomorph of c_0. In this paper, we show that the Orlicz sequence space h_M is isomorphic to a polyhedral Banach space if $\lim_{t\to 0} M(Kt)/M(t) = \infty$ for some $K < \infty$. We also construct an Orlicz sequence space h_M which is c_0-saturated, but which is not isomorphic to any polyhedral Banach space. This shows that being c_0-saturated and having a separable dual are not sufficient for a Banach space to be isomorphic to a polyhedral Banach space.

A Banach space is said to be **polyhedral** if the unit ball of each of its finite dimensional subspaces is a polyhedron. It is **isomorphically polyhedral** if it is isomorphic to a polyhedral Banach space. Fundamental results concerning polyhedral Banach spaces were obtained by Fonf [1, 2].

THEOREM 1 (Fonf): *A separable isomorphically polyhedral Banach space is c_0-saturated and has a separable dual.*

Recall that a Banach space is **c_0-saturated** if every closed infinite dimensional subspace contains an isomorph of c_0. Fonf also proved a characterization of isomorphically polyhedral spaces in terms of certain norming subsets in the dual. In order to state the relevant results, we introduce some terminology due to Rosenthal [4, 5]. The (closed) unit ball of a Banach space E is denoted by U_E.

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Definition: Let E be a Banach space.

(1) A subset $W \subseteq E'$ is **precisely norming** (p.n.) if $W \subseteq U_{E'}$, and for all $x \in E$, there is a $w \in W$ such that $\|x\| = |w(x)|$.

(2) A subset $W \subseteq E'$ is **isomorphically precisely norming** (i.p.n.) if W is bounded and

  (a) there exists $K < \infty$ such that $\|x\| \leq K \sup_{w \in W} |w(x)|$ for all $x \in E$,

  (b) the supremum $\sup_{w \in W} |w(x)|$ is attained at some $w_0 \in W$ for all $x \in E$.

It is easy to see that $W \subseteq E'$ is i.p.n. if and only if there is an equivalent norm $|||\cdot|||$ on E so that W is p.n. in $(E, |||\cdot|||)'$.

THEOREM 2 (Fonf): *Let E be a separable Banach space. Then E is isomorphically polyhedral if and only if E' contains a countable i.p.n. subset.*

This paper is devoted mainly to the problem of identifying the isomorphically polyhedral Orlicz sequence spaces. In §1, a characterization theorem for isomorphically polyhedral Banach spaces having a shrinking basis is proved. This result is applied in §2 to obtain examples of isomorphically polyhedral Orlicz spaces. In §3, a non-isomorphically polyhedral, c_0-saturated Orlicz sequence space is constructed. Since every c_0-saturated Orlicz sequence space has a separable dual, this shows that the converse of Theorem 1 fails, answering a question posed by Rosenthal [4].

Standard Banach space terminology, as may be found in [3], is employed. If (e_n) is a basis of a Banach space E, and $|||\cdot|||$ is a norm on E equivalent to the given norm, we say that (e_n) is **monotone** with respect to $|||\cdot|||$ if $|||\sum_{n=1}^{k} a_n e_n||| \leq |||\sum_{n=1}^{k+1} a_n e_n|||$ for every real sequence (a_n) and all $k \in \mathbb{N}$. Terms and notation regarding Orlicz spaces are discussed in §2.

1. A characterization theorem

This section is devoted to proving the following characterization theorem. Readers familiar with the proofs of Fonf's Theorems will find the same ingredients used here.

THEOREM 3: *Let (e_n) be a shrinking basis of a Banach space $(E, \|\cdot\|)$. The following are equivalent.*

(a) *E is isomorphically polyhedral.*

(b) *There exists an equivalent norm $|||\cdot|||$ on E such that (e_n) is a monotone basis with respect to $|||\cdot|||$, and for all $\sum a_n e_n \in E$, there exists $m \in \mathbb{N}$ such that*
$$|||\sum_{n=1}^{\infty} a_n e_n||| = |||\sum_{n=1}^{m} a_n e_n|||.$$

Proof: Let (P_n) be the projections on E associated with the basis (e_n). The sequence (P_n) is uniformly bounded with respect to any equivalent norm on E. Also, (P_n) converges strongly to the identity operator on E, which we denote by 1. Since (e_n) is shrinking, (P_n') converges to $1'$ strongly as well.

(a) $\Rightarrow$ (b): By renorming, and using Theorem 2, we may assume that E' contains a p.n. sequence (w_k). Fix sequences (ϵ_k) and (δ_k) in $(0,1)$ which are both convergent to 0, and so that $(1+\epsilon_k)(1-2\delta_k) > 1$ for all k. For each k, choose n_k such that $\|(1-P_n)'w_k\| \le \delta_k$ for all $n \ge n_k$. Define a seminorm $|||\cdot|||$ on E by

$$|||x||| = \sup_k (1+\epsilon_k) \max_{1\le n\le n_k} |\langle P_n x, w_k\rangle|. \tag{1}$$

Since $(w_k) \subseteq U_{E'}$, $|||x||| \le 2\|x\| \sup \|P_n\|$. On the other hand, if $x \ne 0$, choose k such that $\|x\| = |w_k(x)|$. Then

$$\begin{aligned}\|x\| &= |w_k(x)| \le |\langle x, P_{n_k}' w_k\rangle| + |\langle x, (1-P_{n_k})'w_k\rangle| \\ &\le |\langle P_{n_k} x, w_k\rangle| + \delta_k\|x\|.\end{aligned}$$

Thus

$$|||x||| \ge (1+\epsilon_k)(1-\delta_k)\|x\| > \|x\|. \tag{2}$$

Hence $|||\cdot|||$ is an equivalent norm on E. It is clear that (e_n) is monotone with respect to $|||\cdot|||$. We claim that this norm satisfies the remaining condition in (b). To this end, we first show that the supremum in the definition (1) is attained. This is trivial if $x = 0$. Fix $0 \ne x \in E$. Choose $k_1 \le k_2 \le \cdots$ and (j_i) , $1 \le j_i \le n_{k_i}$ for all i, so that

$$|||x||| = \lim_i (1+\epsilon_{k_i})|\langle P_{j_i} x, w_{k_i}\rangle|.$$

We divide the proof into cases.

Case 1: $\lim_i k_i = \lim_i j_i = \infty$. In this case, $P_{j_i}x \to x$ in norm. Therefore

$$\limsup_i |\langle P_{j_i} x, w_{k_i}\rangle| = \limsup_i |\langle x, w_{k_i}\rangle| \le \|x\|.$$

Also, $\epsilon_{k_i} \to 0$ as $i \to \infty$. Thus, $|||x||| \le \|x\|$, contrary to (2).

CASE 2: $\lim_i k_i = \infty$, $\lim_i j_i \neq \infty$. By using a subsequence, we may assume that $j_i = j$ for all i. Then

$$|||x||| = \lim_i (1+\epsilon_{k_i})|\langle P_j x, w_{k_i}\rangle| \leq \|P_j x\|.$$

Now choose k such that $\|P_j x\| = |\langle P_j x, w_k\rangle|$. If $j \leq n_k$,

$$\begin{aligned} |||x||| &\geq (1+\epsilon_k)|\langle P_j x, w_k\rangle| \\ &= (1+\epsilon_k)\|P_j x\| > \|P_j x\|, \end{aligned}$$

a contradiction. Now assume $j > n_k$; then

$$\begin{aligned} \|(P_j - P_{n_k})' w_k\| &\leq \|(1-P_j)' w_k\| + \|(1-P_{n_k})' w_k\| \\ &\leq 2\delta_k. \end{aligned}$$

Hence

$$\begin{aligned} \|P_j x\| &= |\langle P_j x, w_k\rangle| \\ &\leq |\langle P_{n_k} x, w_k\rangle| + 2\delta_k \|x\| \\ &\leq (1+\epsilon_k)^{-1}|||x||| + 2\delta_k |||x|||. \end{aligned}$$

Therefore,

$$|||x||| \leq \|P_j x\| \leq ((1+\epsilon_k)^{-1} + 2\delta_k)|||x||| < |||x|||,$$

reaching yet another contradiction. Consequently, we must have

CASE 3: $\lim_i k_i \neq \infty$. By using a subsequence, we may assume that the sequence (k_i) is constant. Then it is clear that the supremum in (1) is attained.

Now for any $x \in E$, choose k so that the supremum in (1) is attained at k. Then it is clear that $|||x||| = |||P_{n_k} x|||$.

(b) $\Rightarrow$ (a): Let (η_n) and (ϵ_n) be sequences convergent to 0, with $1 > \eta_n > \epsilon_n > 0$ for all n. For each n, there is a finite $W_n \subseteq U_{(E,|||\cdot|||)'}$ such that

$$(3) \qquad (1+\epsilon_n)^{-1}|||x||| \leq \max_{w \in W_n} |w(x)| \leq |||x|||$$

for all $x \in \operatorname{span}\{e_1, \ldots, e_n\}$. Define a seminorm ρ on E by

$$(4) \qquad \rho(x) = \sup_n (1+\eta_n) \max_{1 \leq j \leq n} \max_{w \in W_j} |\langle P_j x, w\rangle|.$$

We will show that ρ is an equivalent norm on E, and the set

$$W = \{(1+\eta_n)P_j'w\colon n \in \mathbb{N}, 1 \le j \le n, w \in W_j\}$$

is a countable p.n. subset of $(E,\rho)'$. Then E is isomorphically polyhedral by Fonf's Theorem (Theorem 2). Now let $x \in E$. By (b), there exists m such that $|||x||| = |||P_m x|||$. Hence, by (3), and the fact that (e_n) is monotone with respect to $|||\cdot|||$,

$$\begin{aligned} |||x||| &= |||P_m x||| \\ &\le (1+\epsilon_m) \max_{w \in W_m} |\langle P_m x, w\rangle| \\ &\le (1+\eta_m) \max_{w \in W_m} |\langle P_m x, w\rangle| \\ &\le \rho(x) \\ &\le 2|||x|||. \end{aligned} \tag{5}$$

Thus ρ is an equivalent norm on E. Next we show that the supremum in (4) is attained. Fix $x \in E$. Choose sequences $n_1 \le n_2 \le \cdots$, (j_k), and (w_k) such that $1 \le j_k \le n_k$, $w_k \in W_{n_k}$ for all k, and $\rho(x) = \lim_k (1+\eta_{n_k})|\langle P_{j_k}x, w_k\rangle|$. First assume that $\lim_k n_k = \infty$. Then $\eta_{n_k} \to 0$. Since (e_n) is monotone with respect to $|||\cdot|||$, we have $\rho(x) \le |||x|||$. But there exists k such that $|||x||| = |||P_k x|||$, and there is a $w \in W_k$ such that $|||P_k x||| \le (1+\epsilon_k)|w(P_k x)|$. Thus

$$\rho(x) \ge (1+\eta_k)|w(P_k x)| \ge \frac{1+\eta_k}{1+\epsilon_k}|||x||| > |||x|||,$$

a contradiction. Therefore, $\lim_k n_k \ne \infty$. By going to a subsequence, we may assume that (n_k) is bounded. Using a further subsequence if necessary, we may even assume it is constant. Thus the supremum in (4) is attained. From this it readily follows that the set W is a p.n. subset of $(E,\rho)'$. The countability of W is evident. ∎

Remark: The assumption that the basis (e_n) is shrinking is used only in the proof of (a) ⇒ (b). If (e_n) is assumed to be unconditional and (a) holds, then (e_n) must be shrinking. For otherwise E contains a copy of ℓ^1, which contradicts (a) by Fonf's Theorem (Theorem 1). Thus the assumption of shrinking is not needed if (e_n) is unconditional.

2. Orlicz sequence spaces

In this section, we apply Theorem 3 to identify a class of isomorphically polyhedral Orlicz sequence spaces. Terms and notation about Orlicz sequence spaces follow that of [3]. An **Orlicz function** M is a continuous non-decreasing convex function defined for $t \geq 0$ such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. If $M(t) > 0$ for all $t > 0$, then it is **non-degenerate**. Clearly a non-degenerate Orlicz function must be strictly increasing. The **Orlicz sequence space** ℓ_M associated with an Orlicz function M is the space of all sequences (a_n) such that $\sum M(|a_n|/\rho) < \infty$ for some $\rho > 0$, equipped with the norm

$$\|x\| = \inf\{\rho > 0\colon \sum M(|a_n|/\rho) \leq 1\}.$$

Let e_n denote the vector whose sole nonzero coordinate is a 1 at the n-th position. Then clearly (e_n) is a basic sequence in ℓ_M. The closed linear span of $\{e_n\}$ in ℓ_M is denoted by h_M. Alternatively, h_M may be described as the set of all sequences (a_n) such that $\sum M(|a_n|/\rho) < \infty$ for every $\rho > 0$. Additional results and references on Orlicz spaces may be found in [3]. For a real null sequence (a_n), let (a_n^*) denote the decreasing rearrangement of the sequence $(|a_n|)$.

THEOREM 4: *Let M be a non-degenerate Orlicz function such that there exists a finite number K satisfying $\lim_{t\to 0} M(Kt)/M(t) = \infty$. Then h_M is isomorphically polyhedral.*

Proof: For all $k \in \mathbb{N}$, let

$$b_k = \inf\left\{\frac{M(Kt)}{M(t)}\colon 0 < t \leq M^{-1}\Big(\frac{1}{k}\Big)\right\}.$$

Then $\lim_{k\to\infty} b_k = \infty$. Thus there is a sequence (η_k) decreasing to 1 such that $\eta_k > (1 - b_{k+1}^{-1})^{-1}$ for all k. Define a seminorm on h_M by

$$(6) \qquad |||(a_n)||| = \sup_k \eta_k \|(a_1^*, \ldots, a_k^*, 0, \ldots)\|,$$

where $\|\cdot\|$ is the given norm on h_M. It is clear that $|||\cdot|||$ is an equivalent norm on h_M, and that (e_n) is a monotone basis with respect to $|||\cdot|||$. It suffices to show that $|||\cdot|||$ satisfies the remaining condition in part (b) of Theorem 3. We first show that if (a_n) is a positive decreasing sequence in h_M, then there is a k such that

$$(7) \qquad \|(a_n)\| \leq \eta_k \|(a_1, \ldots, a_k, 0, \ldots)\|.$$

Assume otherwise. There is no loss of generality in assuming that $\|(a_n)\| = 1$. Then $\sum M(a_n) = 1$ and $\sum_{n=1}^{k} M(\eta_k a_n) \leq 1$ for all k. In particular, note that the second condition implies $a_k \leq M^{-1}(1/k)$ for all k, since $\eta_k \geq 1$ and (a_n) is decreasing. Now choose m such that $\|(0,\ldots,0,a_m,a_{m+1},\ldots)\| \leq K^{-1}$. Then $\sum_{n=m}^{\infty} M(Ka_n) \leq 1$. Also $M(Ka_n) \geq b_m M(a_n)$ for all $n \geq m$. Therefore,

$$\begin{aligned} 1 &= \sum M(a_n) \\ &= \sum_{n=1}^{m-1} M(a_n) + \sum_{n=m}^{\infty} M(a_n) \\ &\leq \eta_{m-1}^{-1} \sum_{n=1}^{m-1} M(\eta_{m-1} a_n) + b_m^{-1} \sum_{n=m}^{\infty} M(Ka_n) \\ &\leq \eta_{m-1}^{-1} + b_m^{-1} \\ &< 1, \end{aligned}$$

a contradiction. Hence (7) holds for some k. Now for a general element $(a_n) \in h_M$, choose m such that

$$\|(a_n)\| = \|(a_n^*)\| \leq \eta_m \|(a_1^*,\ldots,a_m^*,0,\ldots)\|.$$

Note that since $\lim_k \eta_k \|(a_1^*,\ldots,a_k^*,0,\ldots)\| = \|(a_n)\|$, the supremum in equation (6) is attained, say, at j. Then choose i large enough that $a_1^*,\ldots,a_j^*$ are found in $\{|a_1|,\ldots,|a_i|\}$. With this choice of i,

$$|||(a_1,\ldots,a_i,0,\ldots)||| \geq \eta_j \|(a_1^*,\ldots,a_j^*,0,\ldots)\| = |||(a_n)|||$$

by choice of j. Since the reverse inequality is obvious,

$$|||(a_n)||| = |||(a_1,\ldots,a_i,0,\ldots)|||,$$

as required. ∎

3. A counterexample

THEOREM 5: *Let M be a non-degenerate Orlicz function. Suppose there exists a sequence (t_n) decreasing to 0 such that*

$$\sup_n \frac{M(Kt_n)}{M(t_n)} < \infty$$

for all $K < \infty$. Then h_M is not isomorphically polyhedral.

Proof: Suppose that h_M is isomorphically polyhedral. By Theorem 3 and the remark following it, one obtains a norm $|||\cdot|||$ on h_M as prescribed by part (b) of the theorem. Fix $\alpha > 0$ so that $|||x||| \leq \alpha \Rightarrow \|x\| \leq 1$. Choose a sequence (η_k) strictly decreasing to 1. Let $n_1 = \min\{n \in \mathbb{N}: \eta_1|||t_n e_1||| \leq \alpha\}$. If $n_1 \leq n_2 \leq \cdots \leq n_k$ are chosen so that $\eta_k|||\sum_{j=1}^k t_{n_j}e_j||| \leq \alpha$, then $\eta_{k+1}|||\sum_{j=1}^k t_{n_j}e_j||| < \alpha$. Hence

$$\{n \geq n_k: \eta_{k+1}|||\sum_{j=1}^k t_{n_j}e_j + t_n e_{k+1}||| \leq \alpha\} \neq \emptyset.$$

Now define

$$n_{k+1} = \min\{n \geq n_k: \eta_{k+1}|||\sum_{j=1}^k t_{n_j}e_j + t_n e_{k+1}||| \leq \alpha\}. \tag{8}$$

This inductively defines a (not necessarily strictly) increasing sequence (n_k) satisfying

$$\eta_k|||\sum_{j=1}^k t_{n_j}e_j||| \leq \alpha \tag{9}$$

for all k and the minimality condition (8). In particular, for all k, $|||\sum_{j=1}^k t_{n_j}e_j||| \leq \alpha$, so $\|\sum_{j=1}^k t_{n_j}e_j\| \leq 1$ by the choice of α. Therefore $\sum_{j=1}^k M(t_{n_j}) \leq 1$ for all k. For all $K < \infty$ and all $k \in \mathbb{N}$,

$$\sum_{j=1}^k M(Kt_{n_j}) \leq \sup_m \frac{M(Kt_m)}{M(t_m)} \sum_{j=1}^k M(t_{n_j}) \leq \sup_m \frac{M(Kt_m)}{M(t_m)}.$$

Consequently, $\sum_{j=1}^\infty M(Kt_{n_j}) < \infty$ for all $K < \infty$. Hence $x = \sum_{j=1}^\infty t_{n_j}e_j$ converges in h_M. Clearly $|||x||| = \lim_k |||\sum_{j=1}^k t_{n_j}e_j||| \leq \alpha$. We claim that in fact $|||x||| = \alpha$. Otherwise, suppose $|||x||| = \beta < \alpha$. Since (e_n) is monotone with respect to $|||\cdot|||$, $|||\sum_{j=1}^k t_{n_j}e_j||| \leq \beta < \alpha$ for all k. By the convergence of x, $\lim_j t_{n_j} = 0$. So one can find i such that $|||t_{n_i}e_j||| \leq \alpha - \beta$ for all j. Then

$$|||\sum_{j=1}^i t_{n_j}e_j + t_{n_i}e_{i+1}||| \leq |||\sum_{j=1}^i t_{n_j}e_j||| + |||t_{n_i}e_{i+1}||| \leq \beta + \alpha - \beta = \alpha.$$

By the minimality condition (8), $n_{i+1} = n_i$. Similarly, we see that $n_j = n_i$ for all $j \geq i$. This contradicts the convergence of x and proves the claim. But now, by (9), $|||\sum_{j=1}^k t_{n_j}e_j||| < \alpha = |||x|||$ for all k, contradicting the choice of the norm $|||\cdot|||$. ∎

We now construct an Orlicz function M satisfying Theorem 5 while h_M is c_0-saturated. We begin with some simple results which help to identify the c_0-saturated Orlicz sequence spaces.

PROPOSITION 6: *Let M be a non-degenerate Orlicz function. Then the following are equivalent.*

(a) *h_M is c_0-saturated;*

(b) *h_M does not contain an isomorph of ℓ^p for any $1 \leq p < \infty$;*

(c) *for all $q < \infty$,*

$$\sup_{0<\lambda,t\leq 1} \frac{M(\lambda t)}{M(\lambda)t^q} < \infty.$$

Proof: Clearly (a) implies (b). If (a) fails, let Y be an infinite dimensional closed subspace of h_M which contains no isomorph of c_0. By [3, Proposition 4.a.7], Y has a subspace Z isomorphic to some Orlicz sequence space h_N. Then h_N contains no isomorph of c_0. By [3, Theorem 4.a.9], h_N contains an isomorph of some $\ell^p, 1 \leq p < \infty$. Hence Y contains a copy of ℓ^p, and (b) fails. The equivalence of (b) and (c) also follows from [3, Theorem 4.a.9]. ∎

PROPOSITION 7: *Let $(b_n)_{n=0}^{\infty}$ be a decreasing sequence of strictly positive numbers such that*

$$\sup_{m,n} \frac{b_{m+n}}{b_n} K^m < \infty \quad \textit{for all } K < \infty.$$

Define M to be the continuous, piecewise linear function such that $M(0) = 0$,

$$M'(t) = \begin{cases} b_n & \textit{if } 2^{-n-1} < t < 2^{-n}, \quad n > 0, \\ b_0 & \textit{if } 2^{-1} < t. \end{cases}$$

Then the Orlicz sequence space h_M is c_0-saturated.

Proof: It is clear that M is a non-degenerate Orlicz function. For all $n \geq 0$, $2^{-n-1}b_n \leq M(2^{-n}) \leq 2^{-n}b_n$. Hence

$$C_q \equiv \sup_{m,n} \frac{M(2^{-m-n})}{M(2^{-n})} 2^{mq} \leq 2 \sup_{m,n} \frac{b_{m+n}}{b_n} (2^{q-1})^m < \infty$$

for any $q < \infty$. Now if $\lambda, t \in (0,1]$, choose $m, n \geq 1$ such that $t \in (2^{-m}, 2^{-m+1}]$, $\lambda \in (2^{-n}, 2^{-n+1}]$. Then $\lambda t \in (2^{-m-n}, 2^{-m-n+2}]$. If $m \geq 2$, then

$$\frac{M(\lambda t)}{M(\lambda)t^q} \leq 2^{2q} \frac{M(2^{-(m-2)-n})}{M(2^{-n})} 2^{(m-2)q} \leq 4^q C_q.$$

If $m = 1$, then $t > 2^{-1}$. Therefore

$$\frac{M(\lambda t)}{M(\lambda)t^q} \leq t^{-q} \leq 2^q.$$

Thus

$$\sup_{0<\lambda,t\leq 1} \frac{M(\lambda t)}{M(\lambda)t^q} < \infty,$$

and h_M is c_0-saturated by the previous proposition. ∎

THEOREM 8: *There exists an Orlicz function M such that h_M is c_0-saturated but not isomorphically polyhedral. In particular, a c_0-saturated space with a separable dual is not necessarily isomorphically polyhedral.*

Proof: It is well known that every c_0-saturated space h_M has a separable dual. Thus the second statement follows from the first. Let $\alpha_0 = \alpha_1 = \alpha_2 = 1$, and let $\alpha_j = (e/j)^j$ for $j \geq 3$. Then (α_j) is a decreasing sequence. Choose a decreasing sequence $(c_j)_{j=0}^{\infty}$ of strictly positive numbers such that $c_{j+1} \leq \alpha_j \alpha_{2j^2} c_j$ for all $j \geq 0$. For convenience, set $s_n = \sum_{j=1}^{n} j$ for all $n \geq 1$. Now define $b_0 = c_0$, $b_1 = c_1$, and $b_{s_n+k} = c_{n+1}/\alpha_{n+1-k}$ whenever $n \geq 1$ and $1 \leq k \leq n+1$. We first show that the sequence (b_j) satisfies the conditions in Proposition 7.

CLAIM 1: (b_j) is a decreasing sequence.
One verifies directly that $b_0 \geq b_1 \geq b_2$. If $n \geq 1$ and $1 \leq k \leq j \leq n+1$,

$$b_{s_n+k} = \frac{c_{n+1}}{\alpha_{n+1-k}} \geq \frac{c_{n+1}}{\alpha_{n+1-j}} = b_{s_n+j}$$

since (α_m) is decreasing. Finally,

$$b_{s_{n+1}+1} = \frac{c_{n+2}}{\alpha_{n+1}} \leq \alpha_{2(n+1)^2} c_{n+1} \leq c_{n+1} = b_{s_n+n+1}$$

for all $n \geq 1$. This proves Claim 1.

CLAIM 2: $b_{m+n} \leq \alpha_m b_n$ for all $m \geq 0$, $n \geq 2$.
Express $n = s_i + k$, $m + n = s_j + l$, where $1 \leq i \leq j$, $1 \leq k \leq i+1$, and $1 \leq l \leq j+1$. If $i = j$, then $l - k = m$. Moreover, $i+1-k \geq \max\{l-k, i+1-l\}$, from which it follows that $\alpha_{i+1-k} \leq \alpha_{l-k}\alpha_{i+1-l}$. Therefore,

$$b_{m+n} = \frac{c_{i+1}}{\alpha_{i+1-l}} \leq \alpha_m \frac{c_{i+1}}{\alpha_{i+1-k}} = \alpha_m b_n.$$

Now consider the possibility that $j > i$. Note first that

$$m = (m+n) - n \leq s_j + j + 1 - (s_i + 1) \leq s_j + j \leq 2j^2.$$

Hence $\alpha_m \geq \alpha_{2j^2}$. Using Claim 1 and the properties of the sequence (c_j), we obtain

$$\begin{aligned} b_{m+n} &\leq b_{s_j+1} = \frac{c_{j+1}}{\alpha_j} \\ &\leq \alpha_{2j^2} c_j \leq \alpha_m c_{i+1} \\ &= \alpha_m b_{s_i+i+1} \leq \alpha_m b_n. \end{aligned}$$

CLAIM 3:

$$\sup_{m,n} \frac{b_{m+n}}{b_n} K^m < \infty \quad \text{for all } K < \infty.$$

First observe that for $i \geq 1$, $1 \leq k \leq i+1$, and $K < \infty$,

$$\begin{aligned} b_{s_i+k} K^{s_i+k} &= \frac{c_{i+1}}{\alpha_{i+1-k}} K^{s_i+k} \leq \alpha_{2i^2} c_i K^{s_i+i+1} \\ &\leq c_0 \alpha_{2i^2} K^{s_i+i+1} \to 0 \end{aligned}$$

as $i \to \infty$. Hence $(b_m K^m)_m$ is bounded. Therefore

$$\sup_{n=1,2} \sup_m b_{m+n} K^m / b_n < \infty.$$

On the other hand, using Claim 2,

$$\sup_{n \geq 2} \sup_m \frac{b_{m+n}}{b_n} K^m \leq \sup_m \alpha_m K^m < \infty$$

by direct verification.

Define the function M using the sequence (b_j) as in Proposition 7. Using Claims 1 and 3, and the proposition, we see that h_M is c_0-saturated. To complete the proof, it suffices to find a sequence (t_n) as in Theorem 5. We claim that the sequence $(t_n) = (2^{-s_n})$ will do. Clearly (t_n) decrease to 0. Fix $m \in \mathbb{N}$. For all $n > m$,

$$b_{s_n - m} = b_{s_{n-1}+(n-m)} = \frac{c_n}{\alpha_m}.$$

Hence

$$\begin{aligned} M(2^m t_n) &= M(2^{-s_n+m}) \leq \frac{b_{s_n-m}}{2^{s_n-m}} \\ &= \frac{c_n}{\alpha_m 2^{s_n-m}} = \frac{2^{m+1}}{\alpha_m} \frac{c_n}{2^{s_n+1}} \\ &= \frac{2^{m+1}}{\alpha_m} \frac{b_{s_n}}{2^{s_n+1}} \leq \frac{2^{m+1}}{\alpha_m} M(t_n) \end{aligned}$$

whenever $n > m$. Therefore,

$$\sup_n \frac{M(2^m t_n)}{M(t_n)} < \infty$$

for all $m \in \mathbb{N}$. ∎

The obvious question to be raised is how to characterize isomorphically polyhedral h_M in terms of the Orlicz function M. We suspect that the condition given in Theorem 4 is the correct one. It can be shown that if

$$\liminf_{t\to 0} M(Kt)/M(t) < \infty \quad \text{for all } K < \infty,$$

then for any sequence (η_k) decreasing to 1, the norm given by equation (6) does not satisfy part (b) of Theorem 3.

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